

Rewriting with *extensionality*

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A brief survey of rewriting
in typed λ -calculi with *extensional* rules

- confluence, decidability and normalization results
- proof techniques
- applications

Extensionality and the lambda calculus

Extensional *axioms* (or equalities) are customary in lambda calculus: they give us

- categoricity (universal property) of the associated data type
- a sort of observational equivalence

Typical examples are:

- η -equality:

$$(\eta) \quad \lambda x.Mx = M \quad (x \notin FV(M))$$

- surjective pairing:

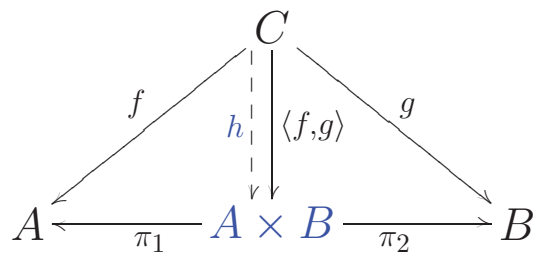
$$(SP) \quad \langle \pi_1(M), \pi_2(M) \rangle = M$$

- case uniqueness:

$$(+!) \quad case(P, M \circ in_A, M \circ in_B) = MP$$

Extensionality and universal properties

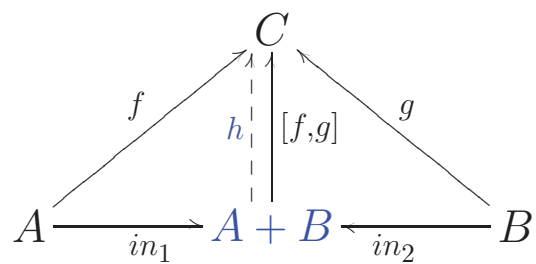
SP



That can be written as:

$$h = \langle f, g \rangle = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$$

Case



That can be written as:

$$h = [f, g] = [h \circ in_1, h \circ in_2]$$

Extensionnality and categories (cont.)

Arrow type and axiom η

$$\begin{array}{ccc} A \times B & \xrightarrow{f} & C \\ \Lambda(f) \times id_B \downarrow & & \nearrow eval \\ C^B \times B & & \end{array}$$

The uniqueness of $h = \Lambda(f)$ can be written

$$\begin{aligned} h &= \Lambda(f) \\ &= \Lambda(eval \circ h \times id_B) \\ &= \Lambda(eval \circ \langle h \circ \pi_1, id_B \circ \pi_2 \rangle) \\ &= \llbracket \lambda x. Mx \rrbracket \text{ if } h = \llbracket M \rrbracket \end{aligned}$$

From equations to rewriting

Two choices to orient extensional equalities:

$$\eta \quad \lambda x.Mx \begin{array}{c} \xrightarrow{\text{green}} \\ \xleftarrow{\text{blue}} \end{array} M \quad x \notin FV(M)$$

$$SP \quad \langle \pi_1(M), \pi_2(M) \rangle \begin{array}{c} \xrightarrow{\text{green}} \\ \xleftarrow{\text{blue}} \end{array} M$$

$$(+!) \quad \text{case}(P, M \circ \text{in}_A, M \circ \text{in}_B) \begin{array}{c} \xrightarrow{\text{green}} \\ \xleftarrow{\text{blue}} \end{array} MP$$

as **contractions**

- + the rules do not depend on types
- the rules are non-local: require search in $FV(M)$ or equality testing
- the rules are not left-linear (except η)
- do not mix well with other rules like Top (lost CR)

as **expansions**

- + the rules are local
- + do mix well with other rules like Top
- depend on types to make sense
- need some restrictions to preserve normalization

Normalization and conditional expansion rules

Two kind of loops can arise using expansions naïvely, let's see the case of η :

Structural:

$$\lambda x.M \longrightarrow \lambda y.(\lambda x.M)y \longrightarrow \lambda y.M[y/x] =_{\alpha} \lambda x.M$$

Contextual:

$$MN \longrightarrow (\lambda x.Mx)N \longrightarrow MN$$

To break the loops we turn expansions into *conditional* rules:

$$(\eta) \quad M \longrightarrow \lambda x : A.Mx \text{ if } \begin{cases} x \text{ fresh} \\ M : A \rightarrow B \\ M \text{ is not a } \lambda\text{-abstraction} \\ M \text{ is not applied} \end{cases}$$

Then, *restricted expansion is no longer a congruence*, but we can show that:

- no equality is lost
- expansions do not introduce new β redexes
- expansion alone converges
- normalization and confluence can be preserved when adding expansions to several calculi

Chronology I

1970s: the first expansion

- 1971 Prawitz suggests to reverse η [Pra71]
- 1976 Huet uses $\beta\eta$ -long normal forms for higher-order unification [Hue76]
- 1979 Mints reverses η and SP [Min79]
- 197- Many people suggest expansions: Martin-Löf, Meyer, Statman, etc.

1980s: the contraction

- 1980 Klop's counterexample to CR for λ +SP [Klo80]
- 1981 Pottinger shows CR for *typed* $\lambda\beta\eta$ +SP [Pot81]
- 1986 Lambek - Scott, Obtulowicz: typed $\lambda\beta\eta$ +SP+**T** is not CR [LS86]
- 1987 Poigné - Voss try completion for $\lambda\beta\eta$ +SP+**T**+sums and recursion [PV87]
- 1989 Nesmith: Klop's counterexample holds for simply typed λ -calculus+fixpoints [Nes89]
- 1991 Curien - Di Cosmo: completion for *polymorphic* $\lambda\beta\eta$ +SP+**T** [CDC96]
- 1994 Necula: η is ok with algebraic non-currified TRS's [Nec94]

Chronology II

1990s: the second expansion

- 1991 Jay: SN for expansions+**T**+**N** [Jay92]
- 1992 Di Cosmo - Kesner: CR+SN for expansions+**T**+sums+weak extensional sums, CR with recursion [DCK93, DCK94b]
- 1992 Cubric: CR for expansions+**T** [Cub92]
- 1992 Ghani - Jay: CR+SN for expansions+**T**+**N** [JG92]
- 1992 Akama: SN+CR for expansions+**T** [Aka93]
- 1992 Dougherty: CR+SN for expansions+**T**+sums, CR with recursion [Dou93]
- 1993 Di Cosmo - Kesner: modularity of CR and SN for expansions + algebraic systems, of CR for recursion [DCK94a]
- 1993 Piperno, Ronchi Della Rocca: expansions for polymorphic type inference [PRDR94]
- 1994 Kesner: CR+SN for pattern calculus with η -expansion [Kes94]
- 1995 Ghani: expansion rules to decide equality for coproducts [Gha95]
- 1995 Di Cosmo - Piperno: SN+CR for polymorphic λ -calculus with η [DCP95]
- 1995 Di Cosmo - Kesner: SN+CR for polymorphic λ -calculus with η, η^2, SP, T via modified reducibility [DCK96]
- 1995 Danvy-Malmkjær-Palsberg: expansions in partial evaluation [DMP95]
- 1996 van Oostrom: CR for untyped η -expansion via developments [vO94]
- 1996 Kesner: η -expansion is the right choice for explicit substitutions [Kes96]
- 1996 Di Cosmo: CR and/or SN for η -expansions in various systems [DC96]
- 1996 Ghani: CR and SN for η -expansions in F^ω [Gha97b]
- 1996 Ghani: CR for η -expansions in *Coc* [Gha97a]
- 1996 Xi: SN for η -expansions in F via internalisation
- 1997 Di Cosmo-Ghani: CR, SN for F^ω with η -expansions and TRSs, CR for *Coc* with η -expansions and TRSs; SN *lost* in *Coc* with usual expansions [DCG97]
- 1999 Barthe: SN for *Coc* with modified expansions, and non-duplicating TRSs [Bar99]

Summary of results

System	Property			
	CR	SN	CR with TRS	CR+SN with TRS
untyped	✓	n.a.	?	n.a.
simply typed	✓	✓	✓	✓
NNO	✓	✓	✓	✓
recursion	✓	n.a.	✓	n.a.
weak case	✓	✓	?	?
strong case	dec.	<i>no?</i>	?	<i>no?</i>
F with η	✓	✓	✓	✓
F with η, η^2, SP	✓	✓	✓	✓
F^ω	✓	✓	✓	✓
LF	?	✓	?	non dupl.
Coc	?	✓	?	non dupl.

Techniques

Many techniques have been used to show SN and/or CR with expansions:

- simulation/interpretation

(Hardin, Tannen, Curien, DiCosmo, Kesner, etc.)

- decomposition

(Akama, DiCosmo, Piperno, Geser, Kahrs, etc.)

- residuals/developements

(Cubric, van Oostrom)

- reductibility/internalisation

(DiCosmo, Kesner, Ghani, Jay, Xi)

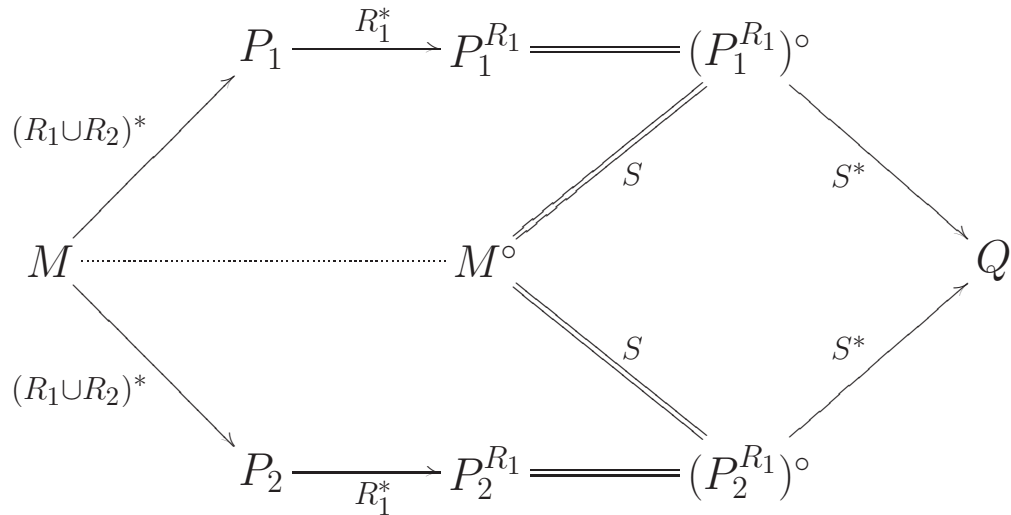
We focus here mainly on some example from the first two classes.

Confluence via simulation/interpretation: a general lemma

Proposition 1.1 (Confluence via simulation) *Given two reduction relations R_1, R_2 on a given set of terms, a reduction $S \subseteq (R_1 \cup R_2)^*$, and a translation $-\circ$ s.t.*

- *if $M \xrightarrow{R_1 \cup R_2} N$ then $M^\circ =_S N^\circ$*
 - *any S divergence on the terms in the image of $R_1 \cup R_2$ via $-\circ$ can be closed using S*
 - *the translation is the identity on the R_1 -normal-forms*
- then if R_1 is weakly normalizing, $R_1 \cup R_2$ is confluent.*

Proof. Here is how to close any divergence of $R_1 \cup R_2$:



□

N.B. for *families* of translations, it suffices to have

$$\text{“ } \forall M^\circ \exists N^\circ. M^\circ =_S N^\circ \text{”}$$

Confluence via simulation/interpretation: a general
lemma, cont'd

One then gets:

- Di Cosmo-Kesner's lemma:

by requiring S to be R_1 , and asking for S reduction instead of equality

- Hardin's lemma:

by requiring R_1 to be SN+CR, using R_1 -n.f. as $_^\circ$, and asking for S reduction instead of equality

- Kamareddine-Rios' lemma:

by using a fixed R_1 -normalization strategy f as $_^\circ$, and asking for S reduction instead of equality

- Kesner's lemma:

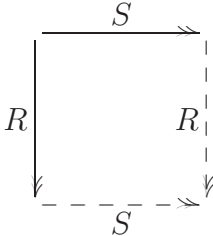
by using R_1 -normalization as $_^\circ$

Decomposition Lemmas

To show that R is CR,

- decompose R into R_1, \dots, R_n
- identify properties of the subsystems s.t. CR for R can be deduced from CR for the R_i 's

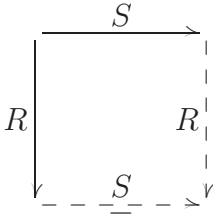
Lemma 1.2 (Hindley-Rosen ([Bar84], section 3))
If \xrightarrow{R} and \xrightarrow{S} are confluent, and commute with each other, i.e.



then $R \cup S$ is confluent.

Establishing the commutation may be complex!

Lemma 1.3 (sufficient condition for commutation)
If

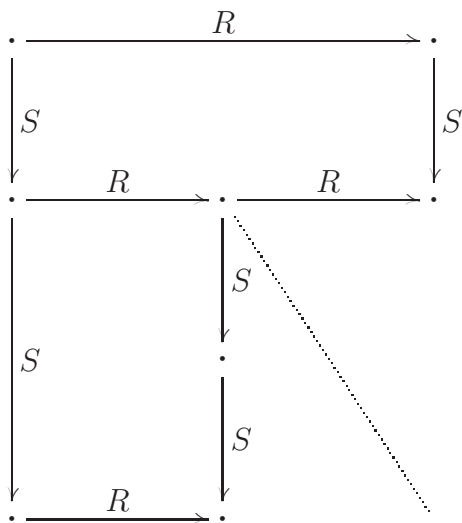


then \xrightarrow{R} and \xrightarrow{S} commute with each other.

Does not work with expansions!

Decomposition Lemmas: DPG

For an *arbitrary* R we cant do better



But if R is a SN, then we can do *different*

Lemma 1.4 (dual condition [Ges90, DCP95])

If R is strongly normalizing, and the following diagram holds

$$(DPG) \quad \begin{array}{ccc} a & \xrightarrow{R} & b \\ \downarrow S & & \downarrow S \\ c & \xrightarrow[-R]{+} & d \end{array}$$

Then \xrightarrow{R} and \xrightarrow{S} commute.

Decomposition Lemmas: Proof of DPG

Proof. Since R is a strongly normalizing rewriting system, we have a well-founded order $<$ on \mathcal{A} by setting $a_1 < a_2$ if $a_2 \xrightarrow{R^*} a_1$. Also, let us denote $dist(a_1, a_2)$ the length of a given S -reduction sequence from a_1 to a_2 . The proof then proceeds by well-founded induction on pairs $(b, dist(a, b))$, ordered lexicographically. Indeed, if b is an R -normal form and $dist(a, b) = 0$, then the lemma trivially holds. Otherwise, by hypothesis, there exist a', a'', a''' as in the following diagram.

$$\begin{array}{ccccc}
 a & \xrightarrow{R} & a' & \xrightarrow{R} & c \\
 \downarrow S & \searrow R & \downarrow S & & \vdots S \\
 a'' & \xrightarrow{R} & a''' & & \\
 \downarrow S & D_1 & \downarrow S & D_2 & \vdots S \\
 b & \xrightarrow{R} & b' & \xrightarrow{R} & d
 \end{array}$$

We can now apply the inductive hypothesis to the diagram D_1 , since

$$(b, dist(a'', b)) <_{lex} (b, dist(a, b)).$$

Finally, we observe that $b \xrightarrow{R^*} b'$, just composing the diagram in the hypothesis down from a .

Hence we can apply the inductive hypothesis to the diagram D_2 , since

$$(b', dist(a', b')) <_{lex} (b, dist(a, b)),$$

and we are done.

□

Rewriting Lemmas: Confluence and Normalization

If one wants both confluence and normalization, here is a nice useful lemma by Akama:

Lemma 1.5 (Akama) *Let R, S be $CR+SN$. If*

$$\begin{array}{ccc}
 M & \xrightarrow{R} & N \\
 \downarrow S & & \downarrow S \\
 M^S & \xrightarrow[\text{+}]{\text{-}R\text{-}} & N^S
 \end{array}$$

then $R \cup S$ is $CR+SN$.

Again, difficult application, so here is how DPG helps:

Lemma 1.6 (Simplified Akama's Lemma) *Let S and R be confluent and strongly normalizing reductions, s.t.*

$$\begin{array}{ccc}
 a & \xrightarrow{R} & c \\
 \downarrow S & & \downarrow S \\
 b & \xrightarrow[\text{+}]{\text{-}R\text{-}} & d
 \end{array}$$

and R preserves S -normal forms: then $S \cup R$ is also confluent and strongly normalizing.

This lemma has been applied in a variety of systems, see [DC96].

Some applications of expansions

higher order unification

here terms are reduced to expansive normal form for unification

decision procedures for category theory

via expansive rewriting systems

isomorphisms of types

there is no nontrivial isomorphism without extensionality and one needs a CR system for studying them, best given with expansions

algebraic functional system

the combination with TRS's leads to problems with contractions, while expansions work fine

Some applications of expansions

pattern calculi

need expansions to rewrite with extensionality [Kes94]

explicit substitutions

extensionality is only reasonable as an expansion: with de Bruijn indexes, testing $x \in FV(M)$ means normalizing the substitution part; while with Delia Kesner showed you can simply write

$$M \rightarrow \lambda. \uparrow (M)1$$

flexible typing

working up to η -expansion can give better typings

partial evaluation

some folklore “tricks” turn out to be expansions

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